Application to areas under curves

- Let \( f(t) \) be a probability density function of a continuous random variable:
  
  \[ f(t) \text{ describes the likelihood for a random variable to take on a given value}. \]

  The probability that \( t \) is between \( a \) and \( b \) \textbf{EQUALS} the area of the shaded region:

  \[
  \Pr[a \leq t \leq b] = \text{Area under } f(t) = \int_a^b f(t) \, dt
  \]

- The most important density function is the \textbf{standard normal distribution} with equation:

  \[
  f(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}
  \]

- To compute the probability that \( t \) is at most some value of \( b \) we have:

  \[
  \Pr[t \leq b] = \int_{-\infty}^b f(t) \, dt
  \]

  As \( b \) varies, this gives a function of \( b \), called the \textbf{cumulative distribution function}.

  Instead of varying \( b \), we use the variable \( x \) in place of \( b \) and denote the integral by \( F(x) \):

  \[
  F(x) = \int_{-\infty}^x f(t) \, dt
  \]

  Note that this is a function written in terms of an integral!
The cumulative distribution function of the standard normal distribution is

\[ F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \, dt \]

and it measures the probability that a random variable will be found at a value \( \leq x \).

**Note:** This integral cannot be computed in terms of elementary functions.

For this reason, we leave the function in an **integral representation**.

To compute various values of \( F(x) \), we can use numerical integration (which is how the bell curve tables were calculated that appear in standard statistics textbooks).

The purpose of the above example is to show that there are disciplines and applications that represent functions as integrals.

**Example**

Let \( F(x) = \int_{0}^{x} (t^2 + 1) \, dt \) be a function of \( x \). Find an alternate representation of \( F(x) \).

**Solution:** We integrate to get:

\[
F(x) = \frac{t^3}{3} + t \bigg|_{0}^{x} = \frac{x^3}{3} + x
\]

Thus, \( F(x) \) is a polynomial of degree 3.

**Note:** \( F'(x) = x^2 + 1 \) which matches the \( t^2 + 1 \) function in the integrand!
The Fundamental Theorem of Calculus - Part I

The first part of the Fundamental Theorem of Calculus tells us how to differentiate certain types of definite integrals and gives a close relationship between integrals and derivatives.

The Fundamental Theorem of Calculus, Part I

If \( f(t) \) is continuous on an interval containing \( a \) then:

\[
\frac{d}{dx} \int_a^x f(t) \, dt = f(x)
\]

When you change the bounds in FTC I, you must apply the chain rule as follows:

\[
\text{FTC I + Chain Rule:} \quad \frac{d}{dx} \int_{u(x)}^{v(x)} f(t) \, dt = f(v(x))v'(x) - f(u(x))u'(x)
\]

FTC I basically says that differentiation and integration are “opposites”.

Example

**Differentiate** \( g(x) \) with respect to \( x \) if \( g(x) = \int_{-2}^{x} \cos(1 + 5t) \sin t \, dt \).

**Solution:** We apply the Fundamental Theorem of Calculus directly to get:

\[
g'(x) = \cos(1 + 5x) \sin x.
\]
Example

Differentiate the following integral with respect to $x$:

$$\int_{10x}^{x^2} t^3 \sin(1 + t) \, dt.$$  

Solution:

- We will use the previous formula: **FTC I + Chain Rule:**

  $$\frac{d}{dx} \int_{u(x)}^{v(x)} f(t) \, dt = f(v(x))v'(x) - f(u(x))u'(x)$$

- We have $f(t) = t^3 \sin(1 + t)$, $u(x) = 10x$ and $v(x) = x^2$.

- Then $u'(x) = 10$ and $v'(x) = 2x$.

- Thus,

  $$\frac{d}{dx} \int_{10x}^{x^2} t^3 \sin(1 + t) \, dt = [(x^2)^3 \sin(1 + (x^2))] [2x] - [(10x)^3 \sin(1 + (10x))] [10]$$

  $$= 2x^7 \sin(1 + x^2) - 10000x^3 \sin(1 + 10x)$$
Example

Differentiate the following integral with respect to $x$:

$$
\int_{x^3}^{2x} 1 + \cos t \, dt
$$

Solution: Using the formula we have:

$$
\frac{d}{dx} \int_{x^3}^{2x} 1 + \cos t \, dt = (1 + \cos(2x))(2) - (1 + \cos(x^3))(3x^2).
$$

Example

Given $F(x) = \int_1^x 2t^2 \, dt$, evaluate $F'(5)$.

Solution: By FTC I we have: $F'(x) = 2x^2$. Hence, $F'(5) = 50$. 
Example

Where is the function $F(x) = \int_0^x (t - 1) \, dt$ increasing and decreasing?

Solution: Recall the increasing/decreasing test:

The Increasing and Decreasing Test

1. If $F'(x) > 0$ on some interval $I$, then $F(x)$ is **increasing** on $I$.
2. If $F'(x) < 0$ on some interval $I$, then $F(x)$ is **decreasing** on $I$.

- We have $F'(x) = x - 1$.
- $F'(x) > 0$ when $x > 1$, thus, $F(x)$ is **increasing** on $(1, \infty)$.
- $F'(x) < 0$ when $x < 1$, thus, $F(x)$ is **decreasing** on $(-\infty, 1)$. 
Example

Evaluate the following limit: \( \lim_{x \to 3^+} \left( \frac{\int_{9}^{x^2} t \ln t \, dt}{x^3 - 27} \right) \).

Solution:

- Plugging in \( x = 3 \) gives a \( 0/0 \) type since \( \int_{9}^{9} t \ln t \, dt = 0 \).
- By L'Hôpital's Rule we have:

\[
\lim_{x \to 3^+} \left( \frac{\int_{9}^{x^2} t \ln t \, dt}{x^3 - 27} \right) \overset{H}{=} \lim_{x \to 3^+} \frac{(x^2 \ln(x^2)) \cdot (2x)}{3x^2}
\]

using FTC I for the derivative of the numerator.

- Simplifying gives the limit to be:

\[
\lim_{x \to 3^+} \frac{2x \ln(x^2)}{3} = 2 \ln 9
\]
Area Between Two Curves

**Problem:**
Given two curves $y = f(x)$ and $y = g(x)$, determine the area enclosed between them.

**Solution:** The area $A$ of the region bounded by the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$ is:

$$A = \int_{a}^{b} |f(x) - g(x)| \, dx.$$  

Informally this can be thought of as follows:

$$\text{Area} = \int_{a}^{b} \text{(top curve)} - \text{(bottom curve)} \, dx, \quad a \leq x \leq b.$$
Area Between Two Curves

Problem:
Given two curves $x = f(y)$ and $x = g(y)$, determine the area enclosed between them.

Solution: The area $A$ of the region bounded by the curves $x = f(y)$ and $x = g(y)$ and the lines $y = c$ and $y = d$ is:

$$A = \int_{c}^{d} |f(y) - g(y)| \, dy.$$ 

Informally this can be thought of as follows:

$$\text{Area} = \int_{c}^{d} (\text{right curve}) - (\text{left curve}) \, dy, \quad c \leq y \leq d.$$
Steps for area between two curves

If you are asked to find the area between two curves, some general guidelines are:

1. Find the intersection points.
2. Draw a sketch of the two curves.
3. Using the sketch determine which curve is the top curve and which curve is the bottom curve (or right curve and left curve).
4. You may need to split the area up into multiple regions.
5. Put the above information into the appropriate formula (once for each region):

   \[ \text{Area} = \int_a^b (\text{top curve}) - (\text{bottom curve}) \, dx, \quad a \leq x \leq b. \]

   \[ \text{Area} = \int_c^d (\text{right curve}) - (\text{left curve}) \, dy, \quad c \leq y \leq d. \]

6. Evaluate the integral using the **Fundamental Theorem of Calculus** (you should get a positive number representing an area)
**Example**

Determine the area enclosed by $y = x^2$ and $y = \sqrt{x}$.

**Solution:**

- **Points of intersection:** $y = x^2$ and $y = \sqrt{x}$ intersect when:
  
  $$x^2 = \sqrt{x} \quad \rightarrow \quad x^4 = x \quad \rightarrow \quad x^4 - x = 0 \quad \rightarrow \quad x(x^3 - 1) = 0$$

  Thus, either $x = 0$ or $x = 1$.

- Sketching the two curves gives:

![Graph showing the area between the curves $y = x^2$ and $y = \sqrt{x}$]

  - The area we want to compute is the shaded region.
  - From the sketch $a = 0$, $b = 1$, the top curve is $y = \sqrt{x}$ and the bottom curve is $y = x^2$.

  $$\text{Area} = \int_a^b [\text{top} - \text{bottom}] \, dx = \int_0^1 (\sqrt{x} - x^2) \, dx = \left( \frac{2}{3}x^{3/2} - \frac{1}{3}x^3 \right) \bigg|_0^1 = \frac{1}{3}.$$
Example

Determine the area enclosed by \( y = x^2, y = \sqrt{x}, x = 0 \) and \( x = 2 \).

Solution:

- From the last problem we have the following sketch:

- Since the top curve changes at \( x = 1 \), we need to use the formula twice.
- For \( A_1 \) we have \( a = 0, b = 1 \), the top curve is \( y = \sqrt{x} \) and the bottom curve is \( y = x^2 \).
- For \( A_2 \) we have \( a = 1, b = 2 \), the top curve is \( y = x^2 \) and the bottom curve is \( y = \sqrt{x} \).

\[
\text{Area} = A_1 + A_2 = \int_0^1 (\sqrt{x} - x^2) \, dx + \int_1^2 (x^2 - \sqrt{x}) \, dx = \frac{1}{3} + \left( \frac{1}{3} x^3 - \frac{2}{3} x^{3/2} \right) \bigg|_1^2
\]

\[
= \frac{1}{3} + \left( \frac{8}{3} - \frac{2(\sqrt{2})^3}{3} \right) - \left( \frac{1}{3} - \frac{2}{3} \right) = \frac{10 - 4\sqrt{2}}{3}
\]
Example

Determine the area enclosed by $y = \sin x$ and $y = \cos x$ on the interval $[0, 2\pi]$.

Solution:

- **Points of intersection:** $y = \sin x$ and $y = \cos x$ intersect when:

$$\sin x = \cos x \quad \rightarrow \quad \tan x = 1 \quad \rightarrow \quad x = \frac{\pi}{4} + \pi k, \ k \text{ an integer}$$

To get this use the unit circle or special triangles:

- Sketching the two curves gives:

Continued on the next slide
Solution (CONTINUED):

- The area we want to compute is the shaded region:

![Graph of cos(x) and sin(x) with shaded region between them.]

- The top curve changes at \( x = \pi/4 \) and \( x = 5\pi/4 \), thus, we need to split the area up into three regions: from 0 to \( \pi/4 \); from \( \pi/4 \) to \( 5\pi/4 \); and from \( 5\pi/4 \) to \( 2\pi \).

\[
\text{Area} = \int_{0}^{\pi/4} (\cos x - \sin x) \, dx + \int_{\pi/4}^{5\pi/4} (\sin x - \cos x) \, dx + \int_{5\pi/4}^{2\pi} (\cos x - \sin x) \, dx
\]

\[
= \left(\sin x + \cos x\right)\bigg|_{0}^{\pi/4} + \left(-\cos x - \sin x\right)\bigg|_{\pi/4}^{5\pi/4} + \left(\sin x + \cos x\right)\bigg|_{5\pi/4}^{2\pi}
\]

\[
= \left(\sqrt{2} - 1\right) + \left(\sqrt{2} + \sqrt{2}\right) + \left(1 + \sqrt{2}\right)
\]

\[
= 4\sqrt{2}
\]
Example

Determine the area enclosed by $x = y^2$ and $x = 8$.

Solution:

- **Points of intersection:** $x = y^2$ and $x = 8$ intersect when:
  
  \[ y^2 = 8 \quad \rightarrow \quad y = \pm \sqrt{8} \quad \rightarrow \quad y = \pm 2\sqrt{2} \]

- Sketching the two curves gives:

  \[ \text{The area we want to compute is the shaded region.} \]
  
  From the sketch $c = -2\sqrt{2}, d = 2\sqrt{2}$, the right curve is $x = 8$ and the left curve is $x = y^2$.

\[
\text{Area} = \int_{c}^{d} \text{[right - left]} \, dy = \int_{-2\sqrt{2}}^{2\sqrt{2}} (8 - y^2) \, dy = \left(8y - \frac{1}{3}y^3\right) \bigg|_{-2\sqrt{2}}^{2\sqrt{2}} \\
= \left[8(2\sqrt{2}) - \frac{1}{3}(2\sqrt{2})^3\right] - \left[8(-2\sqrt{2}) - \frac{1}{3}(-2\sqrt{2})^3\right] = \frac{64\sqrt{2}}{3}
\]
Definite integrals and the Substitution Rule

There are two ways to evaluate a definite integral by substitution.

The first is to evaluate the indefinite integral and then use FTC after.

Another method is to change the limits of integration.

Substitution Rule for Definite Integrals

If $g'$ is continuous on $[a, b]$ and $f$ is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du.$$
Example

Evaluate the following integral: \( \int_{0}^{1} \sqrt{3x + 1} \, dx \).

Solution 1: The first method is to evaluate \( \int \sqrt{3x + 1} \, dx \).

Let \( u = 3x + 1 \), then \( du = 3 \, dx \) and \( dx = \frac{du}{3} \):

\[
\int \sqrt{3x + 1} \, dx = \int \sqrt{u} \frac{du}{3}
\]

\[
= \frac{1}{3} u^{3/2} + C
\]

\[
= \frac{2(3x + 1)^{3/2}}{9} + C.
\]

Then, by FTC II we have:

\[
\int_{0}^{1} \sqrt{3x + 1} \, dx = \left. \frac{2(3x + 1)^{3/2}}{9} \right|_{0}^{1}
\]

\[
= \frac{2(3(1) + 1)^{3/2}}{9} - \frac{2(3(0) + 1)^{3/2}}{9}
\]

\[
= \frac{2^4 - 2}{9} = \frac{14}{9}.
\]
Example

Evaluate the following integral: \( \int_0^1 \sqrt{3x + 1} \, dx \).

Solution 2: The second method is to change the limits.
Let \( u = 3x + 1 \), then \( du = 3 \, dx \).
When \( x = 0 \), then \( u = 3(0) + 1 = 1 \).
When \( x = 1 \) then \( u = 3(1) + 1 = 4 \). Then,

\[
\int_0^1 \sqrt{3x + 1} \, dx = \frac{1}{3} \int_1^4 \sqrt{u} \, du
\]

\[
= \frac{1}{3} \left[ \frac{u^{3/2}}{3/2} \right]_1^4
\]

\[
= \frac{2(4^{3/2})}{9} - \frac{2}{9}
\]

\[
= \frac{14}{9}
\]

Note that we do not return to the variable \( x \) after integrating, we evaluate the expression in \( u \) as the limits of integrations are now in terms of \( u \).
Example
Evaluate \( \int \frac{1}{x^2 + 1} \, dx \).

Solution:
- This is a formula:
\[
\int \frac{1}{x^2 + 1} \, dx = \tan^{-1} x + C
\]

Example
Evaluate \( \int \frac{x}{x^2 + 1} \, dx \).

Solution:
- This is substitution rule. We let \( u = x^2 + 1 \) so that \( du = 2x \, dx \), that is, \( dx = \frac{du}{2x} \):
\[
\int \frac{x}{x^2 + 1} \, dx = \int \frac{u}{2x} \, du = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln |x^2 + 1| + C
\]
Example

Evaluate $\int \frac{1}{x^2 - 2x + 1} \, dx$.

- Factoring the bottom gives $x^2 - 2x + 1 = (x - 1)^2$.
- This problem is a standard substitution problem: $u = x - 1$ and $du = dx$.
- The integral is then:

$$\int \frac{1}{x^2 - 2x + 1} \, dx = \int \frac{1}{(x - 1)^2} \, dx$$

$$= \int (u)^{-2} \, du$$

$$= -u^{-1} + C$$

$$= -\frac{1}{u} + C$$

$$= -\frac{1}{x - 1} + C$$
Examples: Describe how you can solve these problems

- **Example:** \( \int \frac{12}{\sqrt{1 - x^2}} \, dx \).

- **Solution:** You should recognize this as a constant multiple of the following formula:
  \[
  \int \frac{1}{\sqrt{1 - x^2}} \, dx = \sin^{-1} x + C
  \]
  Thus, \( \int \frac{12}{\sqrt{1 - x^2}} \, dx = 12 \sin^{-1} x + C \)

- **Example:** \( \int \sin(7x) \, dx \).

- **Solution:** A simple substitution \( u = 7x \) will work to eventually give \(-\frac{1}{7} \cos(7x) + C\).

- **Example:** \( \int x^7 e^{x^8 + 7} \, dx \).

- **Solution:** Since the derivative of \( x^8 + 7 \) appears (up to a constant) in the question, a substitution will work. Let \( u = x^8 + 7 \) so that \( du = 8x^7 \, dx \) to get:
  \[
  \frac{1}{8} \int e^u \, du = \frac{1}{8} e^u + C = \frac{1}{8} e^{x^8 + 7} + C
  \]
Examples: Describe how you can solve these problems

- **Example:** \( \int (x^2 + 3)\sqrt{x} \, dx \).
  
  **Solution:** Simplify first by expanding products then use the power rule for integration.

- **Example:** \( \int \frac{5}{x^2 + 2x + 2} \, dx \).
  
  **Solution:** We complete the square to get:
  \[
  \int \frac{5}{(x + 1)^2 + 1} \, dx
  \]
  Now it looks like the formula \( \int \frac{1}{1 + x^2} \, dx = \tan^{-1} x + C \).
  Let \( u = x + 1 \) and use substitution rule along with the formula above.

- **Example:** \( \int \cos^9(3x)\tan^6(3x) \, dx \).
  
  **Solution:** Rewriting with sine and cosine we get:
  \[
  \int \cos^9(3x) \frac{\sin^6(3x)}{\cos^6(3x)} \, dx = \int \cos^3(3x)\sin^6(3x) \, dx
  \]
  Now we have a product of sine and cosine. Since cosine is odd, we let \( u = \sin(3x) \).