Do dogs know calculus?

In 2003, Tim Pennings (an Associate Professor at Hope College) performed an experiment with his dog named Elvis.
The Experiment

He took **Elvis** to the beach and threw a ball into the water.

**Question:** What path did Elvis take?
What path did Elvis take?

One possible path is for Elvis to simply jump into the water and swim directly to the ball.

Such a path minimizes distance to the ball.
What path did Elvis take?

But if Elvis doesn’t like the water, he might sprint down the beach to the point on shore closest to the ball and then turn a right angle and swim to the ball.

Such a path minimizes time in the water.
What path did Elvis take?

Surprisingly, Elvis did neither. Instead, Elvis ran part of the way along the beach, and then at some mystery point he plunged into the water swimming diagonally to the ball.

Problem: Why did Elvis take this path? What does this path minimize or maximize?
- Curiously, Elvis took the path that minimizes the time it takes to reach the ball!
- By using Calculus, Prof. Pennings was able to show that Elvis almost always took a path that was close to the optimal solution (he computed the optimal path after performing various measurements of distances and speed).
- Prof. Pennings published his findings in a (now popular) research paper: 
    http://www.maa.org/features/elvisdog.pdf
- His discovery has received international media attention.
- CNN clip on Youtube: http://www.youtube.com/watch?v=yBG8SSB763w
An Example

Problem: Prof. Pennings threw a ball into the water as demonstrated:

To retrieve the ball, Elvis ran 4m along the shoreline and then jumped into the water and swam the rest of the way. Professor Pennings measured Elvis’ running speed and swimming speed and he found that Elvis can run 7 times faster than he can swim (this number is real - check the research paper if you don’t believe me!). Using Calculus, show that the path Elvis took minimizes the time it takes to reach the ball.
Solution

- Let’s start by drawing a picture
- Let point $A$ be where the ball is thrown from
- Let point $B$ be where the ball lands in the water
- Take $C$ to be the point from the ball directly to the shoreline
Let’s assume Elvis runs along the shoreline and at a point $D$, jumps into the water to swim the remaining distance to the ball.

For convenience, we’ll denote the distance from $D$ to $C$ by $x$.

What is the distance from $A$ to $D$?
The distance from $A$ to $D$ is $5 - x$
Summary:
Want to **minimize** time to the ball

\[
\text{speed} = \frac{\text{distance}}{\text{time}} \quad \rightarrow \quad \text{time} = \frac{\text{distance}}{\text{speed}}
\]

Let \( T \) be the time it takes to reach the ball by travelling from \( A \) to \( D \) to \( B \)

**Minimize:**

\[
T = \frac{|AD|}{\text{running speed}} + \frac{|DB|}{\text{swimming speed}}
\]

**Constraints:**

\[
|AD| = 5 - x \quad \text{and} \quad |DB|^2 = x^2 + 7^2
\]

Not given actual speeds. Let’s assume **Elvis** can swim at \( k \) m/s

Hence, **Elvis** can run at \( 7k \) m/s (he runs seven times faster than he swims).

Using that **Elvis** swims at a constant \( k \) m/s and runs at a constant \( 7k \) m/s, we have:

\[
T(x) = \frac{5 - x}{7k} + \frac{\sqrt{x^2 + 49}}{k}
\]

\[
= \frac{5}{7k} - \frac{1}{7k} \cdot x + \frac{1}{k} \cdot \sqrt{x^2 + 49}
\]

In order to minimize \( T(x) \), we find the critical numbers by taking the derivative:

\[
T'(x) = -\frac{1}{7k} + \frac{1}{k} \cdot \frac{1}{2} \cdot (x^2 + 49)^{-1/2}(2x)
\]

\[
= -\frac{1}{7k} + \frac{x}{k \sqrt{x^2 + 49}}
\]
Setting $T'(x) = 0$ and solving we have:

$$T'(x) = -\frac{1}{7k} + \frac{x}{k\sqrt{x^2 + 49}} = 0$$

$$\frac{1}{7k} = \frac{x}{k\sqrt{x^2 + 49}}$$

$$k\sqrt{x^2 + 49} = 7kx$$

Cancelling the $k$'s and squaring gives:

$$\sqrt{x^2 + 49} = 7x \rightarrow x^2 + 49 = 49x^2 \rightarrow 49 = 48x^2 \rightarrow x^2 = \frac{49}{48} \rightarrow x = \frac{7}{4\sqrt{3}} \approx 1.0$$

Since $x = 1.0$ is in the domain of $T(x)$, it is a critical point.

As we have only one critical point, we can use Method 3 to see if it is an absolute minimum.

Taking the second derivative gives: $T''(x) = \frac{49}{k(x^2 + 49)^{3/2}}$, which is always positive.

Hence, by the Second Derivative Test, the absolute minimum occurs at $x = 1.0$ m.

Therefore, the optimal path to minimize the time it takes to the ball is to run

$$5 - 1.0 = 4.0 \text{ m}$$

along the shoreline and then swim the rest of the way.

This is the route that Elvis actually took!